

ON HOMOGENEOUS SPACES AND REDUCTIVE SUBALGEBRAS OF SIMPLE LIE ALGEBRAS

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1. Introduction. Let G be a connected Lie group and H a closed subgroup. Then the homogeneous space $M = G/H$ is called *reductive* if in the Lie algebra $\mathfrak{g}$ of G there exists a subspace $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{m} \dot{+} \mathfrak{h}$ (subspace direct sum) and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ where $\mathfrak{h}$ is the Lie algebra of H (see [4], [5]). In this case the pair $(\mathfrak{g}, \mathfrak{h})$ is called a *reductive pair* and the subspace $\mathfrak{m}$ can be made into an anti-commutative algebra as follows. For $X, Y \in \mathfrak{m}$ let $[X, Y] = XY + \mathfrak{h}(X, Y)$ where $XY = [X, Y]_{\mathfrak{m}}$ (resp. $\mathfrak{h}(X, Y) = [X, Y]_{\mathfrak{h}}$) is the projection of $[X, Y]$ in $\mathfrak{g}$ into $\mathfrak{m}$ (resp. $\mathfrak{h}$). This algebra is related to the canonical G -invariant connection ∇ of the first kind on G/H by $[\nabla_{X^*}(Y^*)]_{P_0} = \frac{1}{2}XY$ where $P_0 = H \in M$ (see [5, Theorem 10.1]).

For a fixed decomposition $\mathfrak{g} = \mathfrak{m} \dot{+} \mathfrak{h}$, the Lie algebra identities of $\mathfrak{g}$ yield the following identities for $\mathfrak{m}$ and $\mathfrak{h}$. For $X, Y, Z \in \mathfrak{m}$ and $U \in \mathfrak{h}$,

- (1) $XY = -YX$ (bilinear);
- (2) $\mathfrak{h}(X, Y) = -\mathfrak{h}(Y, X)$ (bilinear);
- (3) $[Z, \mathfrak{h}(X, Y)] + [X, \mathfrak{h}(Y, Z)] + [Y, \mathfrak{h}(Z, X)] = J(X, Y, Z) \equiv (XY)Z + (YZ)X + (ZX)Y$.
- (4) $\mathfrak{h}(XY, Z) + \mathfrak{h}(YZ, X) + \mathfrak{h}(ZX, Y) = 0$;
- (5) $\mathfrak{h}[(X, Y), U] = \mathfrak{h}([X, U], Y) + \mathfrak{h}(X, [Y, U])$;
- (6) $[U, XY] = [U, X]Y + X[U, Y]$.

In particular (6) says the mappings $\text{ad}_{\mathfrak{m}} U: \mathfrak{m} \rightarrow \mathfrak{m}: X \rightarrow [U, X]$ are derivations of the algebra $\mathfrak{m}$. Using these identities, there was established in [6] a correspondence between simple algebras $\mathfrak{m}$ and holonomy irreducible simply connected spaces G/H which are not symmetric ($\mathfrak{m}\mathfrak{m} = 0$ if and only if G/H is a symmetric space); for example, if G/H is riemannian, then G/H is holonomy irreducible if and only if $\mathfrak{m}$ is a simple algebra.

In this paper, we consider pairs $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is a simple Lie algebra over a field F of characteristic zero and $\mathfrak{h}$ is either semisimple, or regular and reductive (see [2]). In each case we show that the associated $\mathfrak{m}$ is either simple or abelian ($\mathfrak{m}^2 = 0$). This together with [6] shows in particular that if G is a simple connected Lie group and H a closed semisimple or regular reductive Lie subgroup of G such that G/H is simply connected, then either G/H is a symmetric space or G/H is holonomy irreducible. This is a reasonable account of the situation since it can be shown that

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if G/H is a holonomy irreducible pseudo-riemannian reductive space with G simple, then $\mathfrak{h}$ is a reductive subalgebra of $\mathfrak{g}$.

2. The regular reductive case.

LEMMA 1. *Let α be a nonassociative algebra with derivation algebra $\text{Der } \alpha$. Assume that α has no proper ideal stable under $\text{Der } \alpha$. Then either α is simple or $\alpha^2 = 0$.*

Proof. Assume $\alpha^2 \neq 0$ and let $\mathfrak{A}(\alpha)$ denote the associative algebra generated by the left and right multiplications of α [3, p. 290]. Let R be the radical of $\mathfrak{A}(\alpha)$. Then $R\alpha$ is an ideal of α since $\mathfrak{A}(\alpha)(R\alpha) \subseteq (\mathfrak{A}(\alpha)R)\alpha \subseteq R\alpha$. If $D \in \text{Der } \alpha$, then $[D, \mathfrak{A}(\alpha)] \subseteq \mathfrak{A}(\alpha)$ since $\text{ad}_{\text{Hom}(\alpha, \alpha)} D$ stabilizes the set of right and left multiplications (e.g., $[D, L(A)] = L(D(A))$ where $L(B)$ denotes left multiplication by B in α). Thus $\text{ad}_{\mathfrak{A}(\alpha)} D$ is a derivation of $\mathfrak{A}(\alpha)$ and it follows that $[D, R] \subseteq R$ [3, p. 30, exercise 22]. Thus $D(R\alpha) \subseteq [D, R]\alpha + R(D\alpha) \subseteq R\alpha$. Thus $R\alpha$ is a $\text{Der } \alpha$ -stable ideal of α . By assumption, we must have $R\alpha = \alpha$ or $R\alpha = 0$. If $R\alpha = \alpha$, then for some i , $0 = R^i \alpha = R^{i-1} \alpha = \dots = R\alpha = \alpha$ and $\alpha = 0$. Thus we may assume that $R\alpha = 0$. Then $R = 0$ and $\mathfrak{A}(\alpha)$ is completely reducible on α . α^2 is clearly $\text{Der } \alpha$ -stable. Assuming that $\alpha^2 \neq 0$, we must have $\alpha^2 = \alpha$ by hypothesis. We claim that $\alpha^2 = \alpha$ implies that α is simple. For if $\mathfrak{b}$ were a proper ideal of α , then $\mathfrak{b}$ would be $\mathfrak{A}(\alpha)$ -stable and hence $\alpha = \mathfrak{b} \oplus \mathfrak{b}'$ for some $\mathfrak{A}(\alpha)$ -stable $\mathfrak{b}'$. This $\mathfrak{b}'$ would be an ideal and $\alpha = \alpha^2 = \mathfrak{b}^2 + (\mathfrak{b}')^2$ shows that $\mathfrak{b}^2 = \mathfrak{b}$. But then $\mathfrak{b} = \mathfrak{b}^2$ would be $\text{Der } \alpha$ -stable since for B_1, B_2 in $\mathfrak{b}$, $D(B_1 B_2) = (DB_1)B_2 + B_1(DB_2) \in \mathfrak{b}$. Thus α is simple.

We now consider reductive pairs $(\mathfrak{g}, \mathfrak{h})$. Thus let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$, $\mathfrak{m}$ a complementary subspace of $\mathfrak{h}$ in $\mathfrak{g}$ such that $[\mathfrak{m}, \mathfrak{h}] \subseteq \mathfrak{m}$. For $X, Y \in \mathfrak{m}$ we define XY in $\mathfrak{m}$ and $\mathfrak{h}(X, Y)$ in $\mathfrak{h}$ by requiring that $[XY] = XY + \mathfrak{h}(X, Y)$. We regard $\mathfrak{m}$ as a nonassociative algebra with respect to the product XY . Then $\mathfrak{m}$ is clearly anti-commutative and $\text{ad}_{\mathfrak{m}} U$ is a derivation of $\mathfrak{m}$ for $U \in \mathfrak{h}$ (by (6)).

LEMMA 2. *Let $\mathfrak{n}$ be an $\text{ad } \mathfrak{h}$ -stable ideal of $\mathfrak{m}$. Let $\mathfrak{q} = \mathfrak{n} + \mathfrak{h}(\mathfrak{n}, \mathfrak{n})$. If $[\mathfrak{n}, \mathfrak{n}'] \subseteq \mathfrak{q}$ for some complementary subspace $\mathfrak{n}'$ of $\mathfrak{n}$ in $\mathfrak{m}$, then $\mathfrak{q}$ is an ideal of $\mathfrak{g}$.*

Proof. $[\mathfrak{q}, \mathfrak{n}] \subseteq [\mathfrak{n}, \mathfrak{n}] + [\mathfrak{h}(\mathfrak{n}, \mathfrak{n}), \mathfrak{n}] \subseteq \mathfrak{m} + \mathfrak{h}(\mathfrak{n}, \mathfrak{n}) + \mathfrak{n}$ by (3) since $\mathfrak{n}$ is $\text{ad } \mathfrak{h}$ -stable. Thus $[\mathfrak{q}, \mathfrak{n}] \subseteq \mathfrak{q}$. And $[\mathfrak{q}, \mathfrak{h}] \subseteq \mathfrak{q}$ since $\mathfrak{n}$ is $\text{ad } \mathfrak{h}$ -stable and $\mathfrak{q} = \mathfrak{n} + [\mathfrak{n}, \mathfrak{n}]$. It remains to show that $[\mathfrak{q}, \mathfrak{n}'] \subseteq \mathfrak{q}$. But we have

$$\begin{aligned} [\mathfrak{q}, \mathfrak{n}'] &\subseteq \mathfrak{m}\mathfrak{n}' + \mathfrak{h}(\mathfrak{n}, \mathfrak{n}') + [\mathfrak{h}(\mathfrak{n}, \mathfrak{n}), \mathfrak{n}'], \\ [\mathfrak{h}(\mathfrak{n}, \mathfrak{n}), \mathfrak{n}'] &\subseteq [\mathfrak{m}\mathfrak{n}, \mathfrak{n}'] + [[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}'] \subseteq [\mathfrak{n}, \mathfrak{n}'] + [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}']], \\ \mathfrak{h}(\mathfrak{n}, \mathfrak{n}') &\subseteq \mathfrak{m}\mathfrak{n}' + [\mathfrak{n}, \mathfrak{n}']. \end{aligned}$$

But since $[\mathfrak{n}, \mathfrak{n}'] \subseteq \mathfrak{q}$ by hypothesis, $\mathfrak{q}$ contains $[\mathfrak{h}(\mathfrak{n}, \mathfrak{n}), \mathfrak{n}']$ (using (3)) and $\mathfrak{h}(\mathfrak{n}, \mathfrak{n}')$. Since $\mathfrak{m}\mathfrak{n}' \subseteq \mathfrak{n}$ ($\mathfrak{n}$ is an ideal of $\mathfrak{m}$), $[\mathfrak{q}, \mathfrak{n}'] \subseteq \mathfrak{q}$. Thus $\mathfrak{q}$ is an ideal of $\mathfrak{g}$.

LEMMA 3. *Suppose that the Killing form $B(\ , \)$ of $\mathfrak{g}$ is nondegenerate and that $B(\mathfrak{m}, \mathfrak{h}) = 0$. Then $B(\ , \)|_{\mathfrak{m}}$ is nondegenerate and invariant, i.e., $B(XY, Z) = B(X, YZ)$. Moreover every $\text{ad } \mathfrak{h}$ -stable ideal $\mathfrak{n}$ of $\mathfrak{m}$ satisfies $[\mathfrak{n}, \mathfrak{n}^\perp] = 0$ where $\mathfrak{n}^\perp = \{X \in \mathfrak{m} \mid B(X, \mathfrak{n}) = 0\}$.*

Proof. For $X, Y, Z \in \mathfrak{m}$ we have:

$$\begin{aligned} B(XY, Z) &= B([X, Y] - \mathfrak{h}(X, Y), Z) = B([X, Y], Z) = B(X, [Y, Z]) \\ &= B(X, YZ + \mathfrak{h}(Y, Z)) = B(X, YZ). \end{aligned}$$

Now $B(n^\perp, n) = 0$ implies that $0 = B(n^\perp, nm) = B(nn^\perp, m)$. And $B(m, \mathfrak{h}) = 0$ implies that $B(nn^\perp, \mathfrak{h}) = 0$. Thus $B(nn^\perp, g) = 0$ and $nn^\perp = 0$. Consequently $[n, n^\perp] = \mathfrak{h}(n, n^\perp) \subseteq \mathfrak{h}$ and $B([n, n^\perp], m) = 0$. But we also have $B([n, n^\perp], \mathfrak{h}) = B(n^\perp, [n\mathfrak{h}]) = B(n^\perp, n) = 0$. Thus $B([n, n^\perp], g) = 0$ and $0 = [n, n^\perp] = \mathfrak{h}(n, n^\perp)$.

THEOREM 1. *Let $\mathfrak{g}$ be a split simple Lie algebra. Let $\mathfrak{h}$ be a reductive subalgebra of $\mathfrak{g}$ which is normalized by a split Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ (i.e., $\mathfrak{h}$ is reductive and regular [2]). Then $\mathfrak{h}$ has an $\text{ad}(\mathfrak{c} + \mathfrak{h})$ -stable complement $\mathfrak{m}$. Such an $\mathfrak{m}$ is either simple or abelian ($\mathfrak{m}^2 = 0$).*

Proof. We first show that $\mathfrak{c} + \mathfrak{h}$ is reductive. Letting $\mathfrak{g} = \mathfrak{g}_0 + \sum \mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}$, it suffices to show that for $\alpha \neq 0$, $\mathfrak{g}_\alpha \subseteq \mathfrak{c} + \mathfrak{h}$ implies $\mathfrak{g}_{-\alpha} \subseteq \mathfrak{c} + \mathfrak{h}$ [7, p. 669]. Since $[\mathfrak{c}, \mathfrak{h}] \subseteq \mathfrak{h}$ we have $[\mathfrak{c}, \mathfrak{b}] \subseteq \mathfrak{b}$ where $\mathfrak{b}$ is the center of $\mathfrak{h}$. Thus $\mathfrak{c} + \mathfrak{b}$ is solvable. Thus $\text{ad}(\mathfrak{c} + \mathfrak{b})$ is triangulizable and $0 = [\text{ad } \mathfrak{c}, \text{ad } \mathfrak{b}] = \text{ad}[\mathfrak{c}, \mathfrak{b}]$ since $\text{ad}[\mathfrak{c}, \mathfrak{b}] \subseteq \text{ad } \mathfrak{b}$ and $\text{ad } \mathfrak{b}$ consists of semisimple transformations. Thus $[\mathfrak{c}, \mathfrak{b}] = 0$ and $\mathfrak{b} \subseteq \mathfrak{c} = \mathfrak{g}_0$. Now $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{h}^{(1)}$ with $\mathfrak{h}^{(1)}$ semisimple, since $\mathfrak{h}$ is reductive. Let α be a nonzero root such that $\mathfrak{g}_\alpha \subseteq \mathfrak{c} + \mathfrak{h}$. Then since $\mathfrak{h}^{(1)}$ is $\text{ad } \mathfrak{c}$ -stable and $\mathfrak{c} + \mathfrak{h} = \mathfrak{g}_0 + \mathfrak{b} + \mathfrak{h}^{(1)} = \mathfrak{g}_0 + \mathfrak{h}^{(1)}$, we have $\mathfrak{g}_\alpha \subseteq \mathfrak{h}^{(1)}$. Now the restriction of the Killing form $B(\cdot, \cdot)$ of $\mathfrak{g}$ to $\mathfrak{h}^{(1)}$ is nondegenerate since it is the trace form of a faithful representation of the semisimple Lie algebra $\mathfrak{h}^{(1)}$ (see [3, p. 69]). Thus $B(\mathfrak{g}_\alpha, \mathfrak{h}^{(1)}) \neq 0$. Since $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for $\alpha + \beta \neq 0$, it follows $\mathfrak{g}_{-\alpha} \subseteq \mathfrak{h}^{(1)}$. Thus $\mathfrak{g}_\alpha \subseteq \mathfrak{c} + \mathfrak{h}$ implies $\mathfrak{g}_{-\alpha} \subseteq \mathfrak{c} + \mathfrak{h}$ and $\mathfrak{c} + \mathfrak{h}$ is reductive.

It follows that $\mathfrak{h}$ has a complement $\mathfrak{m}$ stable under $\text{ad}(\mathfrak{c} + \mathfrak{h})$. Any complement $\mathfrak{m}$ is the sum of $\mathfrak{m} \cap \mathfrak{g}_0$ and those root spaces $\mathfrak{g}_\beta$ not occurring in $\mathfrak{h}$. In particular, $\mathfrak{g}_\alpha \subseteq \mathfrak{m}$ implies $\mathfrak{g}_{-\alpha} \subseteq \mathfrak{m}$.

We now show that such an $\mathfrak{m}$ is either simple or abelian. Assume that $\mathfrak{m}^2 \neq 0$ and $\mathfrak{m}$ not simple. Then by Lemma 1, $\mathfrak{m}$ has a proper $\text{Der } \mathfrak{m}$ -stable ideal. Since $\mathfrak{m}$ is $\text{ad}(\mathfrak{c} + \mathfrak{h})$ -stable, $\text{ad}(\mathfrak{c} + \mathfrak{h})$ consists of derivations of $\mathfrak{m}$. Thus $\mathfrak{m}$ has a proper ideal $\mathfrak{n}$ stable under $\text{ad}(\mathfrak{c} + \mathfrak{h})$.

Let σ be an automorphism of $\mathfrak{g}$ such that $\sigma|_{\mathfrak{c}} = -\text{id}_{\mathfrak{c}}$ and $\mathfrak{g}_\alpha^\sigma = \mathfrak{g}_{-\alpha}$ for all α (see [3, p. 127]). Then the above discussion shows that $\mathfrak{m}$ and $\mathfrak{h}$ are σ -stable. It follows that $(XY)^\sigma = X^\sigma Y^\sigma$ and $(\mathfrak{h}(X, Y))^\sigma = \mathfrak{h}(X^\sigma, Y^\sigma)$. Thus $\sigma|_{\mathfrak{m}}$ is an automorphism of $\mathfrak{m}$ and $\mathfrak{n}^\sigma$ is an ideal of $\mathfrak{m}$. Since $[\mathfrak{n}^\sigma, \mathfrak{c} + \mathfrak{h}] = [\mathfrak{n}^\sigma, (\mathfrak{c} + \mathfrak{h})^\sigma] = [\mathfrak{n}, \mathfrak{c} + \mathfrak{h}]^\sigma \subseteq \mathfrak{n}^\sigma$, $\mathfrak{n}^\sigma$ is also $\text{ad}(\mathfrak{c} + \mathfrak{h})$ -stable.

Suppose that one of the ideals $\mathfrak{n} \cap \mathfrak{n}^\sigma$, $\mathfrak{n} + \mathfrak{n}^\sigma$ is proper in $\mathfrak{m}$. Call it $\mathfrak{p}$. Then $\mathfrak{p}$ is the sum of $\mathfrak{p} \cap \mathfrak{g}_0$ and root spaces $\mathfrak{g}_\alpha$. Moreover $\mathfrak{g}_\alpha \subseteq \mathfrak{p}$ implies $\mathfrak{g}_{-\alpha} \subseteq \mathfrak{p}$. It follows that $\mathfrak{m} = \mathfrak{m} \cap \mathfrak{g}_0 + \mathfrak{p} + \mathfrak{p}^\perp$ where $\mathfrak{p}^\perp = \{X \in \mathfrak{m} \mid B(X, \mathfrak{p}) = 0\}$ (thus $\mathfrak{g}_\alpha \subseteq \mathfrak{m} - \mathfrak{g}_0$ and $\mathfrak{g}_\alpha \not\subseteq \mathfrak{p}$ implies $\mathfrak{g}_{-\alpha} \not\subseteq \mathfrak{p}$ which implies $B(\mathfrak{g}_\alpha, \mathfrak{p}) = 0$). We use this to show that $\mathfrak{q} = \mathfrak{p} + \mathfrak{h}(\mathfrak{p}, \mathfrak{p})$ is an ideal of $\mathfrak{g}$. By Lemma 2 it suffices to show that $[\mathfrak{p}, \mathfrak{p}'] \subseteq \mathfrak{q}$ where $\mathfrak{p}' = \mathfrak{p}^\perp + \mathfrak{m} \cap \mathfrak{g}_0$. But $[\mathfrak{p}, \mathfrak{m} \cap \mathfrak{g}_0] \subseteq [\mathfrak{p}, \mathfrak{c}] \subseteq \mathfrak{p}$. Thus it suffices to show that $[\mathfrak{p}, \mathfrak{p}^\perp]$

$\subseteq q$. But $B([p, p^\perp], c + \mathfrak{h}) = B(p^\perp, [p, c + \mathfrak{h}]) = B(p^\perp, p) = 0$ and $[p, p^\perp] \subseteq (c + \mathfrak{h})^\perp \subseteq m$. Thus $\mathfrak{h}(p, p^\perp) \subseteq [p, p^\perp] + pp^\perp \subseteq m$ and $\mathfrak{h}(p, p^\perp) = 0$. Thus $[p, p^\perp] = pp^\perp \subseteq p \subseteq q$ and q is an ideal of g . Thus $q = g$ and n cannot be proper in m , a contradiction.

Thus we have $n \cap n^\sigma = 0$ and $n + n^\sigma = m$. Thus $n \cap g_0 = (n \cap g_0)^\sigma = 0$ (since $\sigma|_{g_0} = -\text{id}_{g_0}$). Thus $m \cap g_0 = n \cap g_0 + (n \cap g_0)^\sigma = 0$. It follows that $B(m, \mathfrak{h}) = 0$ (e.g., $m = \sum_{\alpha \in S} g_\alpha$ for some set S of nonzero roots, and $\alpha \in S$ implies $-\alpha \in S$ which implies $g_{-\alpha} \notin \mathfrak{h}$ and therefore $B(g_\alpha, \mathfrak{h}) = 0$). Also $B(n, n) = 0$ (e.g., $n = \sum_{\alpha \in T} g_\alpha$ for some set T of nonzero roots, and $\alpha \in T$ implies $-\alpha \notin T$ which implies $B(g_\alpha, n) = 0$). It follows from Lemma 3 that $[n, n] = nn = \mathfrak{h}(n, n) = 0$. Thus $n^\sigma n^\sigma = 0$. Finally $m^2 = (n + n^\sigma)^2 = n^2 + nn^\sigma + (n^\sigma)^2 \subseteq 0 + n \cap n^\sigma + 0 = 0$, a contradiction.

3. The semisimple case. We now consider the reductive pair $(g, \mathfrak{h})$ where g is a simple Lie algebra and $\mathfrak{h}$ is a semisimple Lie subalgebra. We note that the Killing form $B(\cdot, \cdot)$ of g restricted to $\mathfrak{h}$ is nondegenerate. For if $U, V \in \mathfrak{h}$, then $B(U, V) = \text{tr ad}_g U \text{ad}_g V$ is the trace form of the representation $\text{ad } \mathfrak{h}$ in g , and is nondegenerate by Cartan's criterion [3, p. 69]. (Note that $\text{ad}_g U = 0$ implies $U\mathfrak{f}$ is a one-dimensional ideal in the simple algebra g so that $U = 0$.) Thus if $\mathfrak{h}^\perp = \{X \in g \mid B(X, \mathfrak{h}) = 0\}$, then $\mathfrak{h} \cap \mathfrak{h}^\perp = 0$ and therefore $g = \mathfrak{h}^\perp + \mathfrak{h}$. And $B([\mathfrak{h}^\perp, \mathfrak{h}], \mathfrak{h}) = B(\mathfrak{h}^\perp, [\mathfrak{h}, \mathfrak{h}]) = 0$ so that for $m = \mathfrak{h}^\perp$, $(g, \mathfrak{h})$ is a reductive pair with (fixed) decomposition $g = m + \mathfrak{h}$. Note that since $m = \mathfrak{h}^\perp$, the Killing form B , restricted to m , is a nondegenerate invariant form, i.e., $B(XY, Z) = B(X, YZ)$.

THEOREM 2. *Let g be a simple Lie algebra and $\mathfrak{h}$ a semisimple subalgebra. Then $(g, \mathfrak{h})$ is a reductive pair with $m = \mathfrak{h}^\perp$. Furthermore $m^2 = 0$ or m is simple.*

Proof. Assume $m^2 \neq 0$. Then we have from Lemma 1 that m has a minimal proper $\text{ad } \mathfrak{h}$ -stable ideal n . Then since B is a nondegenerate invariant form on m and $B([XU], Y) = B(X, [UY])$ for $X, Y \in m, U \in \mathfrak{h}$, we have $n^\perp = \{X \in m \mid B(X, n) = 0\}$ is an $\text{ad } \mathfrak{h}$ -stable ideal of m . Thus $n \cap n^\perp$ is an $\text{ad } \mathfrak{h}$ -stable ideal of m ; and since n is minimal, either $n \cap n^\perp = 0$ or $n \cap n^\perp = n$.

In case $n \cap n^\perp = 0$ we have $m = n \oplus n^\perp$. And we know from Lemma 3 that $[n, n^\perp] = 0$. Thus $q = n + \mathfrak{h}(n, n)$ is a proper ideal of g by Lemma 2. This contradiction shows we must have $n \cap n^\perp = n$.

In the case $n \cap n^\perp = n$ we can find an $\text{ad } \mathfrak{h}$ -stable complement, n' (since $\text{ad } \mathfrak{h}$ is semisimple and therefore completely reducible); and we write $m = n + n'$. Thus since $B(n, n) = 0$, to show that $n = 0$ it suffices to show $B(n, n') = 0$.

To find a formula for $B(X, Y)$ with $X, Y \in m$, define $\varepsilon(X)$ and $\delta(X)$ by

$$\varepsilon(X): m \rightarrow \mathfrak{h}: \quad Y \rightarrow \mathfrak{h}(X, Y) \equiv \varepsilon(X)(Y),$$

$$\delta(X): \mathfrak{h} \rightarrow m: \quad U \rightarrow [X, U] \equiv \delta(X)(U),$$

where $U \in \mathfrak{h}$. Using these maps we have for any $Z, X \in m, U \in \mathfrak{h}$ that

$$\begin{aligned} (\text{ad}_g Z)(X) &= [Z, X] = ZX + \mathfrak{h}(Z, X) \\ &= (L(Z) + \varepsilon(Z))(X) \end{aligned}$$

$$(\text{ad}_g Z)(U) = [Z, U] = \delta(Z)(U)$$

and therefore

$$\operatorname{ad}_{\mathfrak{g}} Z = \begin{pmatrix} L(Z) & \varepsilon(Z) \\ \delta(Z) & 0 \end{pmatrix}.$$

From this, note that since $\mathfrak{g}$ is simple $0 = \operatorname{tr} \operatorname{ad}_{\mathfrak{g}} Z = \operatorname{tr} L(Z)$. Also since $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$ is semisimple, and since $\mathfrak{h} \rightarrow \operatorname{ad}_{\mathfrak{m}} \mathfrak{h}: U \rightarrow \operatorname{ad}_{\mathfrak{m}} U$ and $\mathfrak{h} \rightarrow \operatorname{ad}_{\mathfrak{h}} \mathfrak{h}: U \rightarrow \operatorname{ad}_{\mathfrak{h}} U$ are representations of $\mathfrak{h}$, we have $\operatorname{tr} \operatorname{ad}_{\mathfrak{m}} U = \operatorname{tr} \operatorname{ad}_{\mathfrak{h}} U = 0$ for all $U \in \mathfrak{h}$.

Next for $X, Y \in \mathfrak{m}$ define the linear transformation $\sigma(X, Y): \mathfrak{m} \rightarrow \mathfrak{m}$ by $\sigma(X, Y) = \delta(X)\varepsilon(Y)$, that is, $\sigma(X, Y)Z = [X, \mathfrak{h}(Y, Z)] (= [\mathfrak{h}(Z, Y), X])$. From (3) we have the identity

$$\operatorname{ad}_{\mathfrak{m}} \mathfrak{h}(X, Y) - \sigma(X, Y) + \sigma(Y, X) = [L(X), L(Y)] - L(XY)$$

and therefore $\operatorname{tr} \sigma(X, Y) = \operatorname{tr} \sigma(Y, X)$. From this and the matrix for $\operatorname{ad}_{\mathfrak{g}} Z$ we obtain for $X, Y \in \mathfrak{m}$ that

$$\begin{aligned} B(X, Y) &= \operatorname{tr} \operatorname{ad}_{\mathfrak{g}} X \operatorname{ad}_{\mathfrak{g}} Y \\ &= \operatorname{tr} L(X)L(Y) + \operatorname{tr} \varepsilon(X)\delta(Y) + \operatorname{tr} \delta(X)\varepsilon(Y) \\ &= \operatorname{tr} L(X)L(Y) + \operatorname{tr} \delta(Y)\varepsilon(X) + \operatorname{tr} \delta(X)\varepsilon(Y) \\ &= \operatorname{tr} L(X)L(Y) + \operatorname{tr} \sigma(Y, X) + \operatorname{tr} \sigma(X, Y) \\ &= \operatorname{tr} L(X)L(Y) + 2 \operatorname{tr} \sigma(X, Y), \end{aligned}$$

using for the third equality that if $S \in \operatorname{Hom}(V, W)$ and $T \in \operatorname{Hom}(W, V)$ for vector spaces V and W , then $\operatorname{tr} ST = \operatorname{tr} TS$.

Now recall that in the decomposition $\mathfrak{m} = \mathfrak{n} + \mathfrak{n}'$ we must show $B(\mathfrak{n}, \mathfrak{n}') = 0$. Thus for $X \in \mathfrak{n}$, $Y \in \mathfrak{n}'$ we have (from the fact that $\mathfrak{n}$ is an ideal and $\mathfrak{n}\mathfrak{n} = 0$) the matrices

$$L(X) = \begin{pmatrix} 0 & 0 \\ X_{21} & 0 \end{pmatrix} \quad \text{and} \quad L(Y) = \begin{pmatrix} Y_{11} & 0 \\ Y_{21} & Y_{22} \end{pmatrix}$$

and therefore $\operatorname{tr} L(X)L(Y) = 0$ and $B(X, Y) = 2 \operatorname{tr} \sigma(X, Y)$.

To find the matrix for $\sigma(X, Y)$ (with $X \in \mathfrak{n}$, $Y \in \mathfrak{n}'$) let $Z \in \mathfrak{n}$, $Z' \in \mathfrak{n}'$. Then

$$\begin{aligned} \sigma(X, Y)Z &= [\mathfrak{h}(Z, Y), X] \in \mathfrak{n}, \\ \sigma(X, Y)Z' &= [\mathfrak{h}(Z', Y), X] \in \mathfrak{n}. \end{aligned}$$

Therefore

$$\sigma(X, Y) = \begin{pmatrix} \sigma_{11} & 0 \\ \sigma_{21} & 0 \end{pmatrix}$$

and $\operatorname{tr} \sigma(X, Y) = \operatorname{tr} \sigma_{11} = \operatorname{tr}_{\mathfrak{n}} \sigma(X, Y)$. To find the action of $\sigma(X, Y)$ on $\mathfrak{n}$ again let $Z \in \mathfrak{n}$. Then since $\mathfrak{n}$ is an ideal, $\mathfrak{n}\mathfrak{n} = 0$ and $\mathfrak{h}(\mathfrak{n}, \mathfrak{n}) = 0$, we have from (3) that

$$\begin{aligned} 0 &= J(Z, X, Y) = [Z, \mathfrak{h}(X, Y)] + [X, \mathfrak{h}(Y, Z)] \\ &= [-\operatorname{ad}_{\mathfrak{n}} \mathfrak{h}(X, Y) + \sigma(X, Y)]Z. \end{aligned}$$

Therefore on $\mathfrak{n}$ we have $\sigma(X, Y) = \operatorname{ad}_{\mathfrak{n}} \mathfrak{h}(X, Y)$ and since $U \rightarrow \operatorname{ad}_{\mathfrak{n}} U$ is a representation of the semisimple Lie algebra $\mathfrak{h}$, $0 = \operatorname{tr} \operatorname{ad}_{\mathfrak{n}} \mathfrak{h}(X, Y) = \operatorname{tr}_{\mathfrak{n}} \sigma(X, Y)$. Thus $B(\mathfrak{n}, \mathfrak{n}') = 0$ and $\mathfrak{m}$ is simple, a contradiction. Thus either $\mathfrak{m}^2 = 0$ or $\mathfrak{m}$ is simple.

4. **Remarks.** (i) The above discussion for $\mathfrak{h}$ semisimple holds for $\mathfrak{h}$ reductive in $\mathfrak{g}$ except for the assertion that $\text{tr ad}_{\mathfrak{n}} \mathfrak{h}(X, Y) = 0$ and its consequences. The authors do not know whether the theorem holds for all reductive $\mathfrak{h}$.

(ii) If $\mathfrak{h}$ is the zero-space of a derivation of $\mathfrak{g}$ or the one-space of an automorphism of $\mathfrak{g}$, then $\mathfrak{h}$ is reductive and contains a regular element of $\mathfrak{g}$ [1]. Thus if $\mathfrak{g}$ is simple and the underlying field algebraically closed, the associated $\mathfrak{m}$ is simple or abelian by Theorem 1.

(iii) It would be of value to determine all pairs $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ semisimple for which an associated $\mathfrak{m}$ is simple. We now give an example of one nontrivial such pair $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is not simple. Thus let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (direct) where the $\mathfrak{g}_i$ ($i = 1, 2$) are real compact simple Lie algebras. Suppose that $\mathfrak{b}$ is a simple subalgebra of $\mathfrak{g}_1$, $\mathfrak{b}'$ a simple subalgebra of $\mathfrak{g}_2$, $B \rightarrow B'$ an isomorphism from $\mathfrak{b}$ onto $\mathfrak{b}'$. Let $\mathfrak{h} = \{B + B' \mid B \in \mathfrak{b}\}$ and $\mathfrak{m} = \mathfrak{h}^\perp$. Then $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{b}$, and $\mathfrak{b}'$ can easily be chosen such that $\mathfrak{m}^2 \neq 0$. We claim that for any such choice, $\mathfrak{m}$ is simple. By Lemma 1, it suffices to show that $\mathfrak{m}$ has no proper $\text{ad } \mathfrak{h}$ -stable ideal. If $\mathfrak{n}$ were such an ideal, then since the Killing form is negative definite on $\mathfrak{g}$, $\mathfrak{m} = \mathfrak{n} \oplus \mathfrak{n}^\perp$. It is now clear that $\mathfrak{n} + \mathfrak{h}(\mathfrak{n}, \mathfrak{n})$ is an ideal of $\mathfrak{g}$ by Lemma 2, since $[\mathfrak{n}, \mathfrak{n}^\perp] = 0$ by Lemma 3. But then $\mathfrak{n} + \mathfrak{h}(\mathfrak{n}, \mathfrak{n}) = \mathfrak{g}_1$ or $\mathfrak{g}_2$. But by construction, $\mathfrak{h} \cap \mathfrak{g}_1 = \mathfrak{h} \cap \mathfrak{g}_2 = 0$. Thus $\mathfrak{n} = \mathfrak{g}_1$ or $\mathfrak{g}_2$. This is impossible since $B(\mathfrak{n}, \mathfrak{h}) = 0$ whereas $B(\mathfrak{g}_i, \mathfrak{h}) \neq 0$ for $i = 1, 2$.

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